The stability of a viscous fluid between rotating cylinders with an axial flow

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1. Introduction

The stability of viscous flow between rotating cylinders with an axial flow has been investigated theoretically by Goldstein (1937), Chandrasekhar (1960, 1962), and Di Prima (1960); and experimentally by Cornish (1933), Fage (1938), Kaye & Elgar (1957), Donnelly & Fultz (1960) and Snyder (1962*a*). As was pointed out by Di Prima (1960) there were a number of discrepancies in the early work of the 1930's which were clarified in part by the papers of the 1960's. In turn, there appear to be certain small detailed differences in the more recent papers. In part it is these differences with which the present paper is concerned. In addition, the results of the previous theoretical investigations which are limited to the case in which the cylinders rotate in the same direction, are extended to the case of counter rotation.

In the next section the derivation of the eigenvalue problem is briefly sketched, and the differences mentioned above are discussed. The following sections deal with two different methods of solving the eigenvalue problem and the extension to the counter-rotation problem.

2. The eigenvalue problem

Consider two infinitely long cylinders. Let r, θ and z denote the usual cylindrical co-ordinates, and let R_1, R_2, Ω_1 and Ω_2 denote the radii and angular velocities of the inner and outer cylinders, respectively. Let u_r, u_θ and u_z denote the components of velocity in the increasing r, θ and z directions, and p the pressure. The Navier–Stokes equations admit an exact solution of the form

$$\left.\begin{aligned} u_r = 0, \quad u_\theta = V(r), \quad u_z = W(r), \\ \partial p/\partial r = \rho V^2/r, \quad \partial p/\partial z = \text{const.}, \end{aligned}\right\} \tag{1}$$

where ρ is the density.

To consider the stability of this steady motion to rotationally symmetric disturbances we superimpose disturbances of a form such that the θ component of velocity is

$$u_\theta(r,z,t) = V(r) + v(r)\, e^{i(\sigma t+\lambda z)}. \tag{2}$$

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Substituting in the Navier–Stokes equations, neglecting quadratic terms in the disturbance, and making the small gap approximation $d = R_2 - R_1 \ll R_1$, leads to the following system of equations:

$$\{D^2 - a^2 - i\beta + iaRf(x)\}(D^2 - a^2)u + 12iaRu = -a^2T'g(x)v, \tag{3}$$

$$\{D^2 - a^2 - i\beta + iaRf(x)\}v = u, \tag{4}$$

where

$$\left.\begin{aligned} r &= \tfrac{1}{2}(R_1 + R_2) + xd, \quad D = d/dx, \\ \beta &= \sigma d^2/\nu \quad a = \lambda d, \quad R = |W_{av}|\,d/\nu, \quad T' = -4A\Omega_1 d^4/\nu^2, \\ A &= (R_2^2 - R_1^2)^{-1}(\Omega_2 R_2^2 - \Omega_1 R_1^2), \quad \mu = \Omega_2/\Omega_1, \end{aligned}\right\} \tag{5}$$

and

$$f(x) = 6(\tfrac{1}{4} - x^2), \quad g(x) = \tfrac{1}{2}(1+\mu) - (1-\mu)x \tag{6}$$

are the dimensionless axial and circumferential velocities, respectively. Here ν is the kinematic viscosity. The parameter T' which depends upon Ω_1 is called the Taylor number, and R is the Reynolds number associated with the axial velocity. These equations are to be solved subject to the boundary conditions

$$u = Du = v = 0 \tag{7}$$

at $x = \pm\frac{1}{2}$. A more complete derivation of the system of equations (3) and (4) can be found in Chandrasekhar (1961, §79).

The homogeneous system of equations (3) and (4) coupled with the boundary conditions (7) determine an eigenvalue problem. The flow is unstable or stable, according as there are or are not solutions for which the imaginary part of β is negative. We consider only the neutrally stable case, imaginary part of β equal to zero, in which case we obtain a secular equation of the form

$$F(\mu, a, \beta, R, T') = 0. \tag{8}$$

Mathematically the problem is the following: for given real values of μ and R we wish to determine the minimum positive real value of T' with respect to real positive values of a and real values of β. The corresponding values of a and β determine the wavelength and frequency of the disturbance. For $R = 0$, we have the classical Taylor problem with $\beta = 0$, and for $R \neq 0$ our solution will represent Taylor vortices moving in the axial direction with a wave velocity β/aR, based on W_{av}. For small R it appears from the observations of Snyder (1962*a*) that such a motion does occur, but that with increasing R (say $R > 20$) the instability is of a spiral form.

When $\mu \geqslant 0$, the disturbance equations can be further simplified by replacing $g(x)$ by its average value. The error introduced is negligible when $R = 0$, and the computations in §5 indicate this is also true for $R \neq 0$. With this approximation equation (3) reduces to

$$\{D^2 - a^2 - i\beta + iaRf(x)\}(D^2 - a^2)u + 12iaRu = -a^2Tv, \tag{9}$$

where $T = T'\frac{1}{2}(1+\mu)$. The corresponding secular equation is $G(a, R, \beta, T) = 0$. With the further assumption that it is permissible to replace the axial velocity by its average value, i.e. $f(x) = 1$, equations (3) and (4) reduce to a system of equations with constant coefficients. This final problem was solved approximately, but fairly accurately, by Chandrasekhar (1960) and both approximately and exactly by Di Prima (1960). In addition Di Prima considered the significance of averaging the axial flow, and solved the eigenvalue problem with $f(x)$ not averaged by using the Galerkin method. It was found that: (i) in both cases the critical value of T, T_c, increases with increasing R; (ii) for $0 < R < 5$ the variation of T_c with R is essentially the same for both cases; (iii) for $R > 5$, T_c increases more rapidly with R when a parabolic profile is used for $W(r)$; (iv) the wave-number a increases fairly rapidly with R if the average value of $W(r)$ is used, but stays almost constant if a parabolic profile is used; (v) the dimensionless wave velocity β/aR is approximately 0·8 when the average profile is used, and 1·2 when the parabolic profile is used.

The results mentioned above for the variation of T_c with R are in reasonable agreement with the experimental work of Kaye & Elgar (1957) and Donnelly & Fultz (1960). The more recent detailed experiments of Snyder, do, however, raise some questions. First, they do confirm that the correct value of β/aR is approximately 1·2, and that the critical value of a is relatively constant, which shows that it is not permissible to approximate the axial velocity by its average. In addition, however, these experiments indicate that the initial increase of T_c with R is much more rapid than the theoretical predictions (see figure 3, Snyder 1962*a*). For example, with $T_c = 1708$ at $R = 0$, the theoretical value obtained by Di Prima at $R = 2$ is $T_c = 1714$ compared to the experimental value of $T_c = 1835$ at $R = 2{\cdot}2$ which was obtained by Snyder.

Further, Chandrasekhar (1962) using a perturbation procedure, and expanding in powers of $\epsilon = 6aR$ found that $T_c = T_c(R=0) + 26{\cdot}5R^2$ as $R \to 0$, with $a = a(R=0)$. This equation would appear to fit the experimental results of Snyder much better for small R than those found by Di Prima. On the other hand, the fit is not completely satisfactory since the above equation indicates that T_c is parabolic in R, as it must be, while the experiments indicate that T_c varies almost linearly with R as $R \to 0$. These results raise questions concerning the validity of the results obtained by Di Prima (1960) using the Galerkin procedure, and also suggest the necessity for a more detailed analysis of the stability problem.

3. The Galerkin method ($\mu \geqslant 0$)

The eigenvalue problem defined by equations (9), (4) and (7) can be solved approximately by the Galerkin method. In the usual manner, the functions u and v are expanded in complete sets of functions which satisfy the boundary conditions. The secular equation is obtained by requiring that the error in the differential equations be orthogonal to the expansion functions. In the present case the solution can be split into even and odd functions about $x = 0$, the even solution being most critical. In the analysis of Di Prima (1960) the complete orthogonal sets† $C_n(x)$, and $E_n(x) = 2^{\frac{1}{2}} \cos(2n-1)\pi x$ were used for $u(x)$ and $v(x)$, respectively.

In the present analysis a new set of expansion functions are constructed as

† The functions $C_n(x)$ are of the form $(\cosh \lambda_n x)/(\cosh \frac{1}{2}\lambda_n) - (\cos \lambda_n x)/(\cos \frac{1}{2}\lambda_n)$, where the λ_n are the positive roots of $\tanh \frac{1}{2}\lambda + \tan \frac{1}{2}\lambda = 0$. These functions have been tabulated by Reid & Harris (1958).

follows. It is reasonable to expect that the approximating functions can be improved if the effect of the axial flow is incorporated in them. To do this, we consider equations (9) and (4) with $f(x)$ replaced by its average value, and construct the expansion functions $u_{E,m}(x)$ and $v_{C,m}(x)$ as solutions of

$$(D^2-a^2-i\beta+iaR)(D^2-a^2)u_{E,m} = E_m(x), \tag{10}$$

$$(D^2-a^2-i\beta+iaR)v_{C,m} = C_m(x), \tag{11}$$

subject to the boundary conditions (7). In a sense the present scheme is similar to the procedures in vibration problems where the eigenfunction of a complicated

Investigation	R	a_c	β	β/aR	T_c
1	1	3·12	3·65	1·17	1709
2	1	3·12	3·65	1·17	1710
1	2	3·12	7·30	1·17	1713
2	2	3·12	7·30	1·17	1714
1	5	3·12	18·25	1·17	1741
2	5·17	3·12	18·9	1·17	1744
1	10	3·13	36·59	1·17	1840
2	10·34	3·13	37·8	1·17	1852
1	20	3·15	73·43	1·17	2247
2	20·67	3·15	75·8	1·16	2293
1	40	3·22	148·62	1·15	4026
2	40	3·2	147·7	1·15	4066
1	60	3·13	213·26	1·14	7459
2	60	3·15	215·7	1·14	7563

TABLE 1. Critical Taylor numbers and corresponding values of a, and β/aR for assigned values of R. Investigations 1 and 2 refer to the results of the present analysis and those of Di Prima (1960), respectively.

system is expanded in terms of the eigenfunctions of a related simpler system. Here we use approximate eigenfunctions of the system with the axial velocity replaced by its average to solve the same problem with a parabolic profile for the axial velocity.

The functions $u_{E,m}$ and $v_{C,m}$ are complex-valued functions of x, and are clearly rather complicated. They are given in the Appendix. Various inner products which are required in evaluating the secular determinant are recorded in a report by Krueger & Di Prima (1963). Computations have been carried out on the CDC 1604 computer at the University of Wisconsin for a range of values of R up to 60. One-term and two-term series (4×4 determinant) were used for u and v. For fixed R and a, β was chosen so that the value of T for which the secular determinant vanished was real. Then β was varied to find the minimum positive real T, which in turn was minimized over all real positive a. For $R < 20$ the maximum percentage change between the values of T_c for the one- and two-term approximations is less than 8 %. For larger R the change is much greater, for example, at $R = 40$ the change is from 3166 to 4026.

The results using the two-term approximation are tabulated in table 1, along with the results obtained earlier by Di Prima (1960). As can be seen the two sets of results are in very close agreement, particularly so for small R. Thus if the

Galerkin method is used to solve the eigenvalue problem, satisfactory results can be obtained with the simple functions $E_n(x)$ and $C_n(x)$. On the other hand, the present investigation still does not indicate a behaviour of T_c with R as suggested by the perturbation theory of Chandrasekhar (1962). This is considered in the next section. The present results are discussed in more detail in §6, where they are also compared with the experimental work.

4. Fourier series solution

Eliminating u between equations (9) and (4) leads to the sixth-order equation

$$L_0 v = -a^2 T v - i\epsilon L_1 v + \epsilon^2 L_2 v, \tag{12}$$

where

$$\left.\begin{aligned} \epsilon &= 6aR, \\ L_0 &= (D^2-a^2-i\beta)^2(D^2-a^2), \\ L_1 &= (D^2-a^2-i\beta)^2(D^2-a^2)(\tfrac{1}{4}-x^2)+[(\tfrac{1}{4}-x^2)(D^2-a^2)+2](D^2-a^2-i\beta), \\ L_2 &= [(\tfrac{1}{4}-x^2)(D^2-a^2)+2](\tfrac{1}{4}-x^2). \end{aligned}\right\} \tag{13}$$

The boundary conditions are

$$v = D^2 v = D(D^2-a^2-i\beta)v = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{14}$$

Chandrasekhar (1962), neglecting terms $O(\epsilon^2)$, solved equation (12) with the boundary conditions (14) by a perturbation procedure obtaining

$$T_c = 1708 + 26{\cdot}5R^2 \quad \text{and} \quad \beta = 3{\cdot}63R \quad \text{as} \quad R \to 0 \quad \text{for} \quad a = 3{\cdot}1.$$

In this section we will solve the eigenvalue problem defined by equations (12) and (14) with and without terms $O(\epsilon^2)$ by a Fourier series technique.

An appropriate Fourier series for $v(x)$ is

$$v(x) = \sum_{m=1}^{\infty} V_m E_m(x). \tag{15}$$

The boundary conditions $v = D^2 v = 0$ at $x = \pm\frac{1}{2}$ are automatically satisfied; the boundary conditions $D(D^2-a^2-i\beta)v = 0$ at $x = \pm\frac{1}{2}$ introduce the constraint

$$\Gamma = \sum_{m=1}^{\infty} (-1)^m p_m (p_m^2 + a^2 + i\beta) V_m = 0, \tag{16}$$

where $p_m = (2m-1)\pi$. Substituting the series (15) for $v(x)$ and similar series for the higher derivatives of v in equation (12), multiplying the resulting equation by $E_m(x)$, and integrating from $-\frac{1}{2}$ to $\frac{1}{2}$ gives

$$\begin{aligned} &\{(p_m^2+a^2+i\beta)^2(p_m^2+a^2) - a^2T\} V_m \\ &\quad = 2^{\frac{1}{2}} p_m \alpha(-1)^{m+1} + i\epsilon(E_m, L_1 v) - \epsilon^2(E_m, L_2 v) \quad (m = 1, 2, \ldots). \end{aligned} \tag{17}$$

Here $\alpha = 2D^4 v(\frac{1}{2})$ and $(E_m, L_1 v)$ denotes the integral of $E_m L_1 v$ from $-\frac{1}{2}$ to $\frac{1}{2}$.

Since $(E_m, L_1 v)$ and $(E_m, L_2 v)$ involve all of the V_m, equation (17) represents infinitely many equations in infinitely many unknowns. These equations were solved in the following manner. For $\epsilon = 0$, equation (17) can be solved exactly for V_m/α as a function of a, β, and T. Using this set of V_m/α the inner products $(E_m, L_1 v)$ and $(E_m, L_2 v)$ were evaluated, and equation (17) was solved for a new set of V_m/α with $\epsilon \neq 0$. This type of iteration process was carried out a sufficient number of times, and for a sufficiently large m so that Γ could be evaluated accurately. Precisely a sufficient number of terms were taken so that Γ could be evaluated correct to terms $O(10^{-8})$, and T_c was determined to within ± 1. For fixed a, T and β were varied to find the smallest positive real value of T and real β for which equation (16) is satisfied.

The computations are extremely lengthy and for this reason the minimization over a was not carried out. For $a = 3{\cdot}1$, the values of T_c for $R = 1$ and $R = 2$ with and without terms $O(\epsilon^2)$ are given in table 2. Also included are the results of the previous section using the Galerkin method and the results using Chandrasekhar's perturbation theory. The values of β differ only in the third significant figure and hence are not given. The results for the Galerkin method and the Fourier series solution with all terms retained are in excellent agreement. Also it is clear from the results with and without terms $O(\epsilon^2)$ that at $R = 1$ it is not permissible to neglect terms $O(\epsilon^2)$. This shows that the perturbation formula given

	T_c			
R	Perturbation $1708 + 26{\cdot}5\,R^2$	Terms $O(\epsilon^2)$ neglected	Terms $O(\epsilon^2)$ retained	Galerkin $(a = 3{\cdot}12)$
1	1734·5	1737	1710	1709
2	1814	1825	1713	1713

TABLE 2. Critical Taylor number for assigned values of R, $a = 3{\cdot}1$.

by Chandrasekhar cannot be used at $R = 1$. This is not surprising since for $R = 1$, $a = 3{\cdot}1$, the value of ϵ is 18·6.

Consider now the perturbation formula given by Chandrasekhar (1962). This was derived by expanding $\lambda = a^2 T$ in powers of $\epsilon = 6aR$; thus

$$\lambda = \lambda_0(a, \beta) + 6aR\lambda_1(a, \beta) + (6aR)^2 \lambda_2(a, \beta) + \ldots,$$

(see Chandrasekhar 1962: equation (50), with $\sigma = -\beta$). For fixed a, λ_0 and λ_1 were determined as complex-valued functions of β. With the above series terminated after the λ_1 term, the condition imaginary part of $\lambda = 0$ determines β as a function of R. With this value of β, the real part of λ is computed giving the result $T = 1708 + 26{\cdot}5R^2$ for $a = 3{\cdot}1$. However, note that if the $\lambda_2(a, \beta)$ term is retained and if the real part of $\lambda_2(a, \beta)$ starts out as a constant, then this will also contribute to the real part of λ correct to terms $O(R^2)$. Thus to show that the coefficient 26·5 is correct it is necessary to show that the real part of λ_2 does not have a constant term which is real. It is a fairly difficult task to compute the second term in the perturbation series, and this has not been done. However, the present computations using the Fourier series technique do suggest very strongly that the coefficient 26·5 is much too large. This certainly supports the possibility that the real part of λ_2 does start out with a constant term; and hence that the perturbation formula $1708 + 26{\cdot}5R^2$ is not correct even in the limit $R \to 0$.

5. The Galerkin method ($\mu < 0$)

The results of the two previous sections indicate that the eigenvalue problem (9), (4) and (7) can be solved satisfactorily by using the Galerkin method with the simple functions $E_m(x)$ and $C_m(x)$. For $R = 0$ it is also known that the Taylor stability problem can be solved for $-1 < \mu < 1$ by using the Galerkin method with only a few terms of simple expansion functions. Thus it is reasonable to expect that satisfactory results can also be obtained for $R > 0$, $\mu < 0$ by using the Galerkin method.

Consider then the eigenvalue problem defined by equations (3), (4) and (7). The solution can no longer be split into even and odd functions because of the appearance of the linear function $g(x)$. Thus the following series are used for u and v:

$$\left.\begin{aligned} u(x) &= \sum_{m=1}^{M} A_m C_m(x) + \sum_{n=1}^{N} B_n S_n(x), \\ v(x) &= \sum_{m=1}^{M} \alpha_m E_m(x) + \sum_{n=1}^{N} \beta_n F_n(x). \end{aligned}\right\} \tag{18}$$

Here $F_n(x) = 2^{\frac{1}{2}} \sin 2n\pi x$, and the $S_n(x)$ are the odd functions satisfying

$$S_n = DS_n = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2},$$

which have been tabulated by Reid & Harris (1958).

Computations have been carried out for $0 \leqslant R \leqslant 40$ and $-1 \leqslant \mu \leqslant 1$ using the series (18) with $M = 1$, $N = 1$ and $M = 2$, $N = 1$. The results are tabulated in table 3 where T_2' and T_3' indicate the results for $M = 1, N = 1$ and $M = 2$, $N = 1$, respectively. For $R = 0$ the maximum disagreement with the 'exact' results obtained by Chandrasekhar (1954) is $1{\cdot}2\,\%$ at $\mu = -1$. In the range $0 \leqslant R \leqslant 40$, both T_c' and a_c increase with R for all values of μ considered. The ratio of the wave velocity to the average axial velocity β/aR, decreases slightly with increasing R and decreasing μ. This effect is most noticeable for $\mu = -1$. The variation of T_c' with R for different values of μ is in general agreement with some unpublished experiments of Snyder (1962*b*) except for the differences in the manner in which T_c' behaves as $R \to 0$. These differences are the same as those mentioned earlier for $\mu = 0$.

For $\mu > -0{\cdot}25$ the results with $\Omega(r)$ replaced by its average value are still in close agreement with those obtained using a linear profile. Even at $\mu = -0{\cdot}5$, $R = 20$, the percentage difference based on T_c' is only $6{\cdot}7\,\%$.

6. Conclusions

In figure 1 the variation of T_c (the eigenvalue problem (9), (4), (7)) with R, for $R \to 0$, for the various computations are shown. Also shown are the experimental points of Snyder (1962*a*) for $\mu = 0$ and some recent unpublished experiments of Schwarz, Springett & Donnelly (1963) for $\mu = 0$. As has been mentioned earlier the solution of the perturbation equations with and without terms $O(\epsilon^2)$ by the Fourier series technique would appear to confirm the correctness of the results obtained using the Galerkin method. The discrepancy between the experimental results of Snyder and Schwarz *et al.* is not yet understood.

μ	R	a	β/aR	T_3'	T_3'/T_2'
1	0	3·11	—	1,709	0·99
	1	3·11	1·17	1,710	0·99
	5	3·11	1·17	1,742	1·0
	10	3·15	—	1,842	1·02
	20	3·15	—	2,255	1·10
	40	3·20	1·15	4,068	1·14
0·5	0	3·11	—	2,277	0·99
	1	3·11	1·17	2,279	0·99
	5	3·11	1·17	2,321	1·0
	10	3·13	1·17	2,455	1·02
	20	3·15	1·17	3,005	1·10
	40	3·17	1·15	5,419	—
0·25	0	3·11	—	2,727	—
	1	3·13	1·17	2,730	—
	5	3·13	1·17	2,780	—
	10	3·13	1·17	2,940	—
	20	3·15	1·17	3,598	—
	40	3·17	1·15	6,485	—
0	0	3·12	—	3,392	0·99
	1	3·13	1·17	3,395	0·99
	5	3·13	1·17	3,458	0·99
	10	3·13	1·16	3,675	1·02
	20	3·17	1·16	4,472	1·10
	40	3·20	1·15	8,049	—
−0·25	0	3·15	—	4,465	0·99
	1	3·15	1·17	4,468	0·99
	5	3·15	1·17	4,550	1·0
	10	3·15	1·17	4,809	1·02
	20	3·20	1·16	5,873	1·03
	40	3·20	1·15	10,515	—
−0·5	0	3·15	—	6,420	0·99
	1	3·17	1·17	6,424	0·99
	5	3·20	1·17	6,538	1·0
	10	3·22	1·16	6,901	1·01
	20	3·25	1·16	8,384	1·03
	40	3·30	1·13	14,639	—
−0·75	0	3·40	—	10,529	0·99
	1	3·40	1·15	10,536	0·99
	5	3·40	1·15	10,706	1·0
	10	3·45	1·15	11,237	1·01
	20	3·50	1·14	13,334	1·03
	40	3·70	1·11	20,733	—
−1·0	0	4·0	—	18,895	0·99
	1	4·0	1·14	18,902	—
	5	4·0	1·11	19,090	—
	10	4·0	1·10	19,662	—
	20	4·1	1·10	21,746	—
	40	4·2	1·07	27,866	—

TABLE 3. Critical Taylor numbers T and corresponding values of a and β/aR for assigned values of μ and R.

The differences between the present theoretical results and the experimental results of Schwarz *et al.* can be explained in part by correcting for gap size. The experiments of Schwarz *et al.* were run for a value of $R_1/R_2 = 0{\cdot}945$. It has been shown by Walowit, Tsao & Di Prima (1964) that for $R = 0$ and $R_1/R_2 = 0{\cdot}95$,

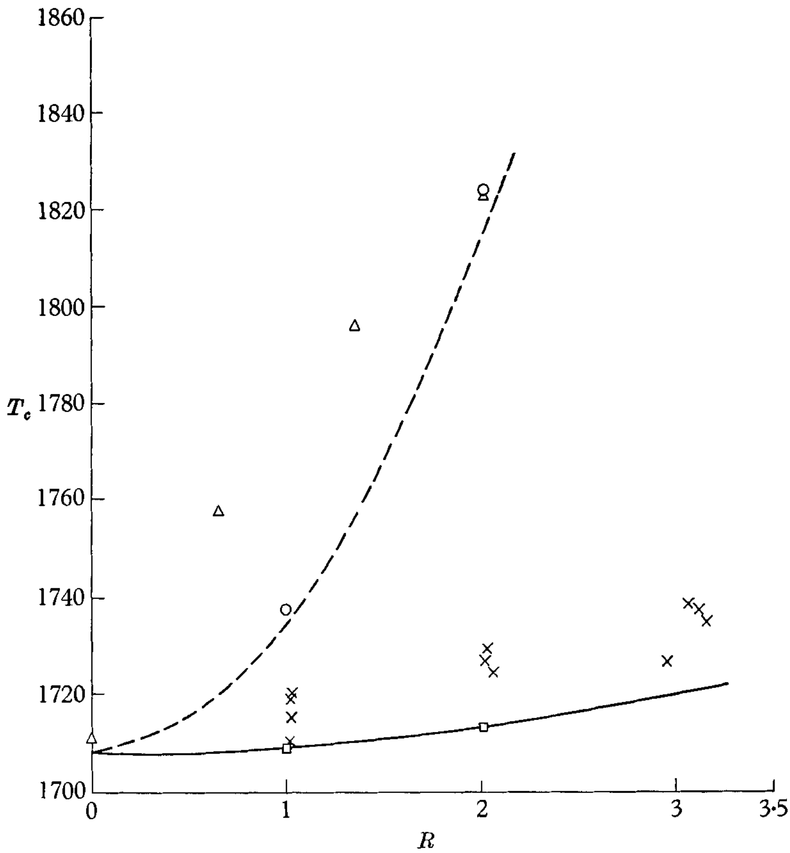

FIGURE 1. The variation of the critical Taylor number T_c with the axial Reynolds number R as $R \to 0$. The solid curve refers to the present results using the Galerkin method. The short dashed curve refers to the perturbation results of Chandrasekhar (1962). The Fourier series results with terms $O(\epsilon^2)$ neglected and retained are denoted by ○'s and □'s respectively. The experimental results of Snyder (1962*a*) for $\mu = 0$, $R_1/R_2 = 0{\cdot}948$ and Schwarz, Springett & Donnelly (1963) for $\mu = 0$, $R_1/R_2 = 0{\cdot}945$ are denoted by △'s and ×'s, respectively.

$T_c = 1755$ in contrast to a value of 1708 for $R_1/R_2 \to 1$. Thus for this geometry the correction for gap size when $R = 0$ is 46. Although this correction is not necessarily independent of R, it is interesting to note the results when the theoretical curve of T_c as a function of R is corrected by this amount. The curves with and without the correction for gap size are shown in figure 2. They essentially bracket the experimental results of Schwarz *et al.* Also shown are the experimental results of Snyder.

The variation of wave number with R is shown in figure 3. The results are in good agreement with those of Snyder for $R < 20$. It should be remembered that

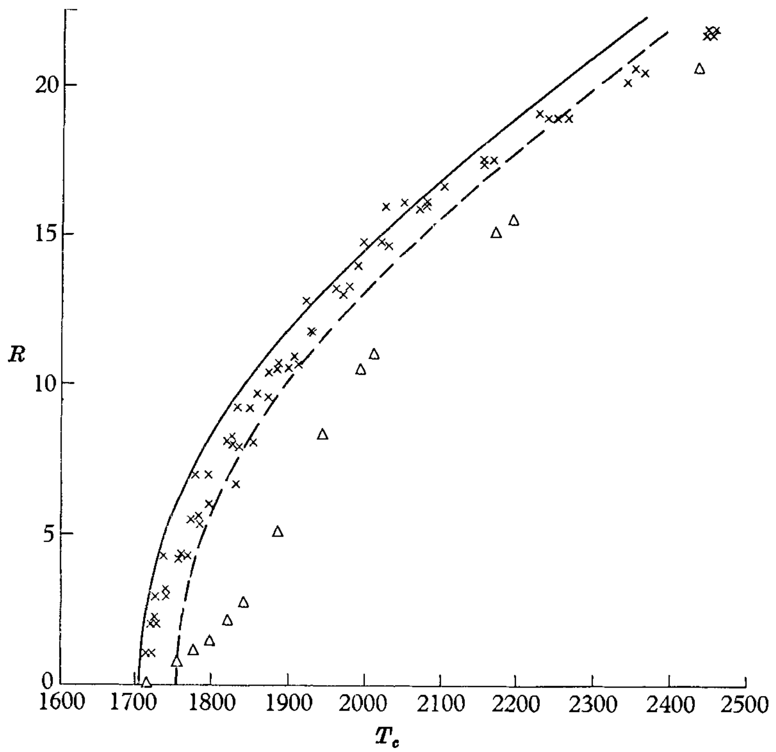

FIGURE 2. The variation of the critical Taylor number T_c with the axial Reynolds number R. The solid curve refers to the present results using the Galerkin method. The dashed curve refers to these results corrected for a gap size $R_1/R_2 = 0{\cdot}95$. The experimental results of Snyder (1962*a*) for $\mu = 0$, $R_1/R_2 = 0{\cdot}948$ and Schwarz, Springett & Donnelly (1963) for $\mu = 0$, $R_1/R_2 = 0{\cdot}945$ are denoted by $\triangle$'s and $\times$'s respectively.

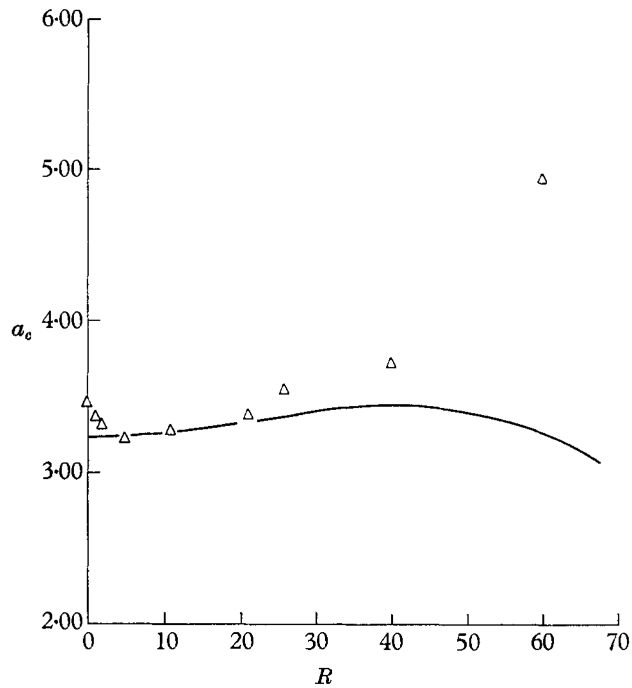

FIGURE 3. The variation of the critical wave number a_c with the axial Reynolds number R for $\mu \geqslant 0$. The experimental results of Snyder (1962*a*) for $\mu = 0$, $R_1/R_2 = 0{\cdot}948$ are indicated by $\triangle$'s.

for R greater than approximately 20 the instability, as reported by Synder, is of a spiral form rather than of the type considered here. The results for the wave velocity are also in good agreement with the measurements of Snyder, both results indicating that the wave velocity is a slowly decreasing function of R, being about $1{\cdot}17W_{av}$ at $R = 1$.

For $\mu < 0$, the results are qualitatively the same as for $\mu \geqslant 0$.

Finally, the present results appear to show that the stability of flow between rotating cylinders to rotationally symmetric disturbances can be solved in a satisfactory manner by using the Galerkin method.

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Appendix

The $u_{E,m}$ and $v_{C,m}$ functions which satisfy equations (10) and (11) and the boundary conditions (7) are

$$u_{E,m}(x) = \frac{2^{\frac{1}{2}}p_m}{\delta_{2m-1}}\left\{\frac{(-1)^{m+1}}{Z}\left[\frac{\cosh\xi_1 x}{\cosh\frac{1}{2}\xi_1} - \frac{\cosh\xi_2 x}{\cosh\frac{1}{2}\xi_2}\right] + \frac{\cos p_m x}{p_m}\right\},$$

$$v_{C,m}(x) = \frac{1}{\phi_m}\left\{(D^2+q^2)\,C_m(x) - 2\lambda_m^2\frac{\cosh qx}{\cosh\frac{1}{2}q}\right\},$$

where $\delta_{2m-1} = (p_m^2+a^2)(p_m^2+q^2)+12iaR, \quad Z = \xi_1\tanh\frac{1}{2}\xi_1 - \xi_2\tanh\frac{1}{2}\xi_2,$

$$\phi_m = \lambda_m^4 - q^4, \quad p_m = (2m-1)\pi, \quad q^2 = a^2 + i(\beta - aR),$$

and $\xi_1, \xi_2, -\xi_1, -\xi_2$ are the roots of the equation

$$(\xi^2-a^2)(\xi^2-q^2)+12iaR = 0.$$

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